Mass Energy Relation of the Nonlinear Spinor

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Abstract

The nonlinear spinor fields coupled with the interactive vector and scalar fields are important in the research of the elementary particles. The analysis of this paper shows that the different kind of fields results in different energy-speed relation, the mass-energy relation $E = mc^2$ exactly holds only for the linear part of the coupled fields. The specific energy-speed relations may be useful to identify the concrete particle and interaction models via elaborated experiments.

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1 Introduction

The Einstein's mass-energy relation

$$E = mc^2, \qquad m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{1.1}$$

is the apotheosis of the elegance and simplicity of modern science, which becomes one of the cornerstone of the modern physics. However its origin is mainly relevant to the linear theories. In [1] we find that the energy caused by nonlinear potential terms slightly violates (1.1), $E = mc^2$ is mainly valid for linear part of the field equations.

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The effects caused by the nonlinear part continuously depend on the moving speed of the particle, and the different kind of nonlinear potential corresponds to different function of speed v. These functions may be detected by elaborated experiments, which inversely can give a definite identification for the models of interactive fields.

In this paper we take $\hbar = c = 1$ as unit. Denote the Minkowski metric by $\eta_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$, Pauli matrices by

$$\vec\sigma = (\sigma^j) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{1.2}$$

Define 4×4 Hermitian matrices as follows

$$\alpha^\mu = \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & \vec\sigma \\ \vec\sigma & 0 \end{pmatrix} \right\}, \ \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \tag{1.3}$$

We adopt the Hermitian matrices (1.3) instead of Dirac matrices γ^μ for the convenience for calculation.

In [1], we considered nonlinear spinors with self electromagnetic field, the corresponding Lagrangian is given by

$$\mathcal{L}_e = \phi^+[\alpha^\mu(\hbar i\partial_\mu - eA_\mu) - \mu\gamma]\phi - \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + F, \tag{1.4}$$

where $\mu > 0$ is a constant mass, $F(\check\gamma) > 0$ is the nonlinear function of the quadratic scalar $\check\gamma \equiv \phi^+\gamma\phi$.

For the following nonlinear potential

$$F = \frac{1}{2}w\check\gamma^2, \qquad w > 0, \tag{1.5}$$

making classical approximation, we derived the corresponding mass-energy relation, or more accurately say, the energy-speed relation as follows[1]

$$\begin{cases} p^\mu = \left(m_e + W\ln\frac{1}{\sqrt{1-v^2}}\right)u^\mu, \quad W \equiv \frac{1}{2}w\int\check\gamma^2 d^3\bar x, \\ E_e = \frac{m_e}{\sqrt{1-v^2}} + W\left(\frac{1}{\sqrt{1-v^2}}\ln\frac{1}{\sqrt{1-v^2}} + \sqrt{1-v^2}\right). \end{cases} \tag{1.6}$$

where $\bar x$ stands for the central reference coordinate system of the electron, and W is the proper energy corresponding to nonlinear term, m_e the static mass, u^μ the 4-vector speed. In (1.6) the energy caused by self electromagnetic field is not included, where it was treated as independent field variables.

In what follows we derive the complete energy-speed relation for several nonlinear potential and interactive fields, which are used frequently in particle models. The method is in principle similar to [1], so some trivial derivations are overleaped.

2 Classical Mechanics and Mass-Energy Relation

In this paper we investigate from the following Lagrangian,

$$\begin{aligned}\mathcal{L} = \;& \phi^+\alpha^\mu(i\partial_\mu - eA_\mu)\phi - \mu\check{\gamma} + F(\check{\gamma}) - s\check{\gamma}G \\ & -\tfrac{1}{2}\kappa(\partial_\mu A_\nu \partial^\mu A^\nu - a^2 A_\mu A^\mu) - \tfrac{1}{2}\lambda(\partial_\mu G\partial^\mu G - b^2G^2),\end{aligned} \tag{2.1}$$

where A^μ and G include the self and external potential, $\kappa = \pm 1$ and $\lambda = \pm 1$ are used to stand for the repulsive or attractive seif interaction, e.g. A^μ stands for repulsive electromagnetic potential if $\kappa = 1, a = 0$, but stands for attractive interactive potential similar strong interaction if $\kappa = -1$. G stands for a scalar interactive potential similar Higgs field, which is repulsive if $\lambda = -1$, and attractive if $\lambda = 1$. $F(\check{\gamma}) > 0$ is a concave function satisfying

$$F'(\check{\gamma})\check{\gamma} > F(\check{\gamma}), \quad \text{for} \quad (\check{\gamma} > 0). \tag{2.2}$$

In (2.1), the interactive potentials are representative, which need not exactly stand for some known or unknown fields. What important is that different kind of field will results in different energy-speed relation which can be tested by experiments.

The corresponding dynamic equation is given by

$$\alpha^\mu(i\partial_\mu - eA_\mu)\phi = (\mu + sG - F')\gamma\phi, \tag{2.3}$$

$$(\partial_\alpha\partial^\alpha + a^2)A^\mu = \kappa e q^\mu, \tag{2.4}$$

$$(\partial_\alpha\partial^\alpha + b^2)G = \lambda s\check{\gamma}. \tag{2.5}$$

The Hamiltonian form of (2.3) reads

$$\hbar i\partial_t\phi = \hat{H}\phi, \quad \hat{H} = eA_0 + \vec{\alpha}\cdot(-i\nabla - e\vec{A}) + (\mu + sG - F')\gamma. \tag{2.6}$$

The complete dynamic equation of A^μ is given by the following Maxwell equation[2]

$$\begin{cases} \partial_\mu q^\mu = \partial_\mu A^\mu = 0, & (\partial^\alpha\partial_\alpha + a^2)A^\mu = \kappa e q^\mu, \\ \vec{E} = -\nabla A^0 - \partial_0\vec{A}, & \vec{B} = \nabla\times\vec{A}, \\ \nabla\cdot\vec{E} = \kappa e q^0 - a^2A^0, & \nabla\times\vec{E} = -\partial_0\vec{B}, \\ \nabla\cdot\vec{B} = 0, & \nabla\times\vec{B} = \partial_0\vec{E} + \kappa e\vec{q} - a^2\vec{A}, \end{cases} \tag{2.7}$$

where $\vec{A} = (A^1, A^2, A^3)$ is the spatial part of a contravariant vector A^μ. For the scalar G we have not similar decomposition with manifest physical meanings.

By (2.6) we have the current conservation law

$$\partial_\mu q^\mu = 0, \qquad q^\mu = \phi^+\alpha^\mu\phi, \tag{2.8}$$

so we can take the normalizing condition as

$$\int_{R^3} q^0 d^3x = \int_{R^3} |\phi|^2 d^3x = 1. \tag{2.9}$$

Assume (2.3) has particle-like solution, we define some classical concepts such as coordinate, speed, momentum, energy etc. as follows[1]

Definition 1. *The coordinate $X(t)$ and speed v of the particle (2.3) are defined respectively by*

$$\vec{X}(t) \equiv \int_{R^3} \vec{x}|\phi|^2 d^3x = \int_{R^3} \vec{x} q^0 d^3x, \quad \vec{v} \equiv \frac{d}{dt}\vec{X}. \tag{2.10}$$

Lemma 1. *By (2.10) and the current conservation law $\partial_\mu q^\mu = 0$, we have the speed of the particle*

$$\vec{v} = \int \vec{x}\partial_0 q^0 d^3x = -\int \vec{x}(\nabla\cdot\vec{q})d^3x = \int \vec{q} d^3x. \tag{2.11}$$

By(2.9) and (2.11), we have the following classical approximation

$$q^\mu \to v^\mu \delta(\vec{x} - \vec{X}), \qquad v^\mu = (1, \vec{v}). \tag{2.12}$$

Definition 2. *Define the 4-dimensional momentum p^μ and energy E of the particle or system (2.1) respectively by*

$$p^\mu \equiv \int_{R^3} \phi_k^+ (i\partial^\mu - eA^\mu)\phi d^3x, \tag{2.13}$$

$$E \equiv \int_{R^3} \left(\sum_{\forall f} \frac{\partial \mathcal{L}}{\partial(\partial_t f)} \partial_t f - \mathcal{L} \right) d^3x = p^0 + E_F + E_A + E_G, \tag{2.14}$$

$$E_F = \int_{R^3} (F'\check{\gamma} - F) d^3x, \tag{2.15}$$

$$E_A = -\frac{1}{2}\kappa \int_{R^3} (\partial_0 \boldsymbol{A}_\mu \partial^0 \boldsymbol{A}^\mu + \nabla \boldsymbol{A}_\mu \cdot \nabla \boldsymbol{A}^\mu + a^2 \boldsymbol{A}_\mu \boldsymbol{A}^\mu - 2\kappa e q_0 \boldsymbol{A}^0) d^3x, \tag{2.16}$$

$$E_G = -\frac{1}{2}\lambda \int_{R^3} (\partial_0 \boldsymbol{G} \partial^0 \boldsymbol{G} + \nabla \boldsymbol{G} \cdot \nabla \boldsymbol{G} + b^2 \boldsymbol{G}^2) d^3x, \tag{2.17}$$

where $\boldsymbol{A}^\mu$ and $\boldsymbol{G}$ are potentials produced by the spinor ϕ itself, which satisfy the natural boundary condition.

In the central reference coordinate system $\bar{x}^\mu$ of the spinor, since the spinor takes the energy eigenstate, we have $\partial_0 \mathbf{A}^\mu = \partial_0 \mathbf{G} = 0$. By the Green functions of (2.4) and (2.5), we have the corresponding static energy

$$W_F = \int_{R^3} (F'\check{\gamma} - F) d^3\bar{x} > 0, \tag{2.18}$$

$$W_A = \frac{1}{2}\kappa \int_{R^3} (\mathbf{A}_\mu (\Delta - a^2)\mathbf{A}^\mu + 2\kappa e q_0 \mathbf{A}^0) d^3\bar{x} \tag{2.19}$$

$$
\begin{aligned}
&= \frac{1}{2}e\int_{R^3}[q_0\mathbf{A}^0+\vec{q}\cdot\vec{\mathbf{A}}]d^3\bar{x}\\
&= \frac{\kappa e^2}{8\pi}\int_{R^6}\frac{e^{-ar}}{r}[|\phi(\bar{x})|^2|\phi(\bar{y})|^2+\vec{q}(\bar{x})^2\cdot\vec{q}(\bar{y})^2]d^3\bar{x}d^3\bar{y}\\
&\doteq \frac{\kappa e^2}{8\pi}\int_{R^6}\frac{e^{-ar}}{r}|\phi(\bar{x})|^2|\phi(\bar{y})|^2d^3\bar{x}d^3\bar{y},\\
W_G &= \frac{1}{2}\lambda\int_{R^3}\mathbf{G}(\Delta-b^2)\mathbf{G}d^3\bar{x}=-\frac{1}{2}s\int_{R^3}\check{\gamma}\mathbf{G}d^3\bar{x} \qquad (2.20)\\
&= -\frac{\lambda s^2}{8\pi}\int_{R^6}\frac{e^{-br}}{r}\check{\gamma}(\bar{x})\check{\gamma}(\bar{y})d^3\bar{x}d^3\bar{y},
\end{aligned}
$$

where $r=|\bar{x}-\bar{y}|$. By (2.20) and (2.21), we find W_A, W_G provide contrary self energy, so the scalar is quite different from the vector field at this point.

Make Lorentz transformation, we have $d^3x=\sqrt{1-v^2}d^3\bar{x}$, we get energy-speed relation for each part of a moving spinor as follows

$$
\begin{aligned}
E_F &= W_F\sqrt{1-v^2}, && (2.21)\\
E_A &\doteq W_A\left(\frac{2}{\sqrt{1-v^2}}-\sqrt{1-v^2}-\frac{2v^2}{3\sqrt{1-v^2}}\right)+W_a\frac{v^2}{\sqrt{1-v^2}}, && (2.22)\\
E_G &= W_G\left(\sqrt{1-v^2}+\frac{2v^2}{3\sqrt{1-v^2}}\right)-W_b\frac{v^2}{\sqrt{1-v^2}}. && (2.23)
\end{aligned}
$$

where

$$
W_a=\frac{\kappa}{3}\left(\frac{ae}{4\pi}\right)^2\int_{R^3}\left(\int_{R^3}\frac{e^{-ar}}{r}|\phi(\bar{y})|^2d^3\bar{y}\right)^2d^3\bar{x},
$$

$$
W_b=\frac{\lambda}{3}\left(\frac{bs}{4\pi}\right)^2\int_{R^3}\left(\int_{R^3}\frac{e^{-br}}{r}\check{\gamma}(\bar{y})d^3\bar{y}\right)^2d^3\bar{x}.
$$

The detailed transformation laws will be discussed elsewhere.

Now we examine the term p^μ. It is easy to check

Lemma 2. *For any Hermitian operator $\hat{P}$, and corresponding classical quantity P for e by*

$$
P\equiv\int_{R^3}\phi^+\hat{P}\phi d^3x, \qquad (2.24)
$$

then we have

$$
\frac{d}{dt}P=\int_{R^3}\phi^+\left(\partial_t\hat{P}+i[\hat{H},\hat{P}]\right)\phi d^3x, \qquad (2.25)
$$

where $[\hat{H},\hat{P}]=\hat{H}\hat{P}-\hat{P}\hat{H}$.

By lemma 2, we have

Theorem 1. *For 4-d momentum p^μ, we have the following rigorous dynamic equation*

$$
\begin{cases}
\frac{d}{dt}p^0=\int(e\vec{q}\cdot\vec{E}+s\check{\gamma}\partial_0G)d^3x-\frac{d}{dt}E_F,\\
\frac{d}{dt}\vec{p}=\int[e(q^0\vec{E}+\vec{q}\times\vec{B})-s\check{\gamma}\nabla G]d^3x,
\end{cases} \qquad (2.26)
$$

where $\vec{E}$ and $\vec{B}$ include the intensity of self field.

Making classical approximation (2.12), we get the Newton's second law

$$\begin{cases} \frac{d}{dt}p^0 = e\vec{v}\cdot\vec{E}(t,\vec{X}) + sW_\gamma\sqrt{1-v^2}\partial_t G(t,\vec{X}) - \frac{d}{dt}E_F, \\ \frac{d}{dt}\vec{p} = e(\vec{E}+\vec{v}\times\vec{B}) - sW_\gamma\sqrt{1-v^2}\nabla G(t,\vec{X}), \end{cases} \tag{2.27}$$

where $W_\gamma = \int_{R^3}\check{\gamma}d^3\bar{x}$.

By the dynamic equation (2.3), we have[1]

$$p^\mu = \int_{R^3}\mathrm{Re}[\phi^+\alpha^\mu(i\partial_0 - eA_0)\phi]d^3x. \tag{2.28}$$

Assume the spinor takes the energy eigenstate and moves smoothly, by (2.12) and (2.28) we find $p^\mu \propto v^\mu$, then by covariance we have

$$p^\mu = mu^\mu, \tag{2.29}$$

where $m = \sqrt{p^\mu p_\mu}$ is the scalar mass of the spinor. (2.29) times (2.27) and using (2.12), we have

$$\frac{1}{2}\frac{d}{dt}(p^\mu p_\mu) = msW_\gamma v^\mu\partial_\mu G - mW_F\frac{d}{dt}\ln\sqrt{1-v^2},$$
$$\frac{d}{dt}m = \frac{d}{dt}[sW_\gamma G - W_F\ln(\sqrt{1-v^2})].$$

so we can define the inertial mass of the spinor as

$$m = m_0 + sW_\gamma G(t,\vec{X}) + W_F\ln\frac{1}{\sqrt{1-v^2}}, \tag{2.30}$$

where m is the moving inertial mass, and m_0 is static mass. Substituting (2.30) into (2.29) we get

$$p^\mu = \left(m_0 + sW_\gamma G + W_F\ln\frac{1}{\sqrt{1-v^2}}\right)u^\mu. \tag{2.31}$$

Substituting (2.31), (2.21), (2.22) and (2.23) into (2.14) we finally get the complete energy-speed relation for spinor ϕ.

3 Discussion and Conclusion

Now we extract some important information from the above analyses:

(C1). The numerical results[3, 4, 5, 6, 7, 8, 9] showed that the mass contributed by potentials W_F, W_A, W_G and so on are much less than $m_0 \approx \mu$, so the Einstein's mass-energy relation holds with high precision generally.

(C2). By (2.31), (2.21), (2.22) and (2.23), we learn different potential has different energy-speed relation, so we can identify each interacting field by testing the fine structure of the energy-speed relation of a particle.

(C3). By (2.30) or (2.31) we find the external scalar G manifestly influence the inertial mass and momentum of a particle. This effect will seriously ruin the classical mechanics, so the scalar field should be absent in the Nature[2, 7, 10].

(C4). The Lagrangian for vector field A^μ have two different formalism due to the symmetry of the Maxwell equation, i.e.,

$$\mathcal{L}_1 = -\frac{1}{2}\kappa(\partial_\mu A_\nu \partial^\mu A^\nu - a^2 A_\mu A^\mu), \quad \mathcal{L}_2 = \frac{1}{2}(\vec{E}^2 - \vec{B}^2). \tag{3.1}$$

$\mathcal{L}_1$ seems to be more definite and natural than $\mathcal{L}_2$.

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